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# Linear Form of 3-scale Special Relativity Algebra and the Relevance of Stability

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**ABSTRACT:** We show that the algebra of the recently proposed Triply Special Relativity can be brought to a linear (*i.e.*, Lie) form by a correct identification of its generators. The resulting Lie algebra is the stable form proposed by Vilela Mendes a decade ago, itself a reappearance of Yang's algebra, dating from 1947. As a corollary we assure that, within the Lie algebra framework, there is no Quadruply Special Relativity.

# 1 Introduction

During the last few years, a significant number of papers has focused on the question of deforming the Lie algebra of standard quantum relativistic kinematics. By the latter we refer to the Poincaré algebra, extended by the promotion of the coordinates  $X^\mu$  to generator status. The main motivation behind this effort is the inference of an algebraic signature of quantum gravity. Accordingly, the first step taken, is the postulation of the Heisenberg commutation relations among  $X^\mu$  and the momenta  $P_\mu$ , while the four-vector nature of the former is retained. The new invariant scale introduced as a result of the deformation (in addition to those given by the velocity of light and Planck's constant), is generally identified with the Planck scale. The resulting spacetime is non-commutative, the characteristic length of the non-commutativity given by the Planck length. Quite recently, an additional deformation was proposed [8], that, roughly speaking, does the same to the momentum sector. The new (fourth) invariant scale introduced is related to the cosmological constant, and shows up in the non-commutativity of the momenta.

In the initially proposed deformations, the commutators of the generators were expressed as general analytic functions of the same, rather than linear ones, *i.e.*, the Lie algebra framework was abandoned. It was later realized that, by suitable (nonlinear) redefinitions of the generators, linearity could be restored, *i.e.*, the proposed deformations lived, after all, within the (suitably completed) universal enveloping algebra of a Lie algebra (albeit, in some cases, with a different coproduct). In the recent additional deformation mentioned above, the imposition of the Jacobi identities results, according to the authors of [8], to non-linearity of the  $x - p$  commutation relations. Since the rest of the commutation relations are linear, it is not at all clear that this non-linearity can be also made to disappear by a redefinition of the generators. This would seem to suggest that the introduction of this new invariant scale in Special Relativity can only be achieved at the cost of parting with the familiar, and powerful, Lie algebra machinery. Our first aim in this note is to point out that this is not the case. A proper identification of the generators permits to close the algebra linearly. The point, we think, is particularly important. It shows that *invariant scales can be consistently introduced in Special Relativity within the Lie algebra framework*. Our wider, and, by far, most important, aim though is to bring to the attention of the authors in the field the relevance of the formal Lie algebra deformation theory and the related concept of Lie algebra stability. With this in mind, we comment on the similarities between the non-linear algebras that have been proposed and the stable Lie algebra put forth by Vilela Mendes in [9] more than a decade ago<sup>2</sup>.

Before delving into these matters in detail, we wish to make two comments. The first one concerns the scope of the present work. As a rule, the above mentioned deformations are Hopf algebra deformations. In this short note, we deal exclusively with their algebraic sector — a thorough analysis of their full Hopf algebra structure, as well as a detailed presentation of the Lie algebra stability point of view and the physical interpretation of the resulting stable algebra will be the subject of a longer article, presently in preparation.

The second comment concerns nomenclature. The generally used term for this field is “Doubly Special Relativity” (DSR) — recently promoted to “Triply”. As explained in sufficient detail in [1], this is a misnomer. The “Special” of “Special Relativity” refers to the restriction on the coordinate transformations considered, not to the existence of an invariant scale, therefore, the multiple-scale relativities proposed are as (singly) Special as the original one. We think the above term is conceptually inappropriate enough to warrant its abolishment, and propose as an alternative the “ $n$ -scale Special Relativity” ( $n$ -SSR) of the title. As explained below,  $n=0,1,2,3$  are the only possibilities within the Lie algebra framework. We emphasize at this point that  $n > 1$  does not imply non-linearity — non-linear deformations will be referred to explicitly, *e.g.*, the algebra of [8] is a non-linear 3-SSR.

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<sup>2</sup>This algebra has actually first appeared in the work of Yang [13] — see Sect. 4

## 2 The Linear Form of “Triply Special Relativity”

The construction of the deformed algebra in [8] starts by attempting to put together a non-commutative spacetime with a similarly non-commutative energy-momentum space. The commutation relations initially postulated are

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} \\ [M_{\mu\nu}, X_\rho] &= -g_{\mu\rho}X_\nu + g_{\nu\rho}X_\mu \end{aligned} \quad (1)$$

$$[M_{\mu\nu}, P_\rho] = -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu \quad (2)$$

$$[X_\mu, X_\nu] = \frac{1}{\kappa^2}M_{\mu\nu} \quad (3)$$

$$[P_\mu, P_\nu] = \frac{1}{R^2}M_{\mu\nu}, \quad (4)$$

and the question is asked whether the Jacobi identities can be satisfied. The conclusion reached is that the canonical commutation relations,

$$[X_\mu, P_\nu] = g_{\mu\nu}, \quad (5)$$

have to be deformed. We read:

*One finds by explicit computation that the Jacobi identities are satisfied if one takes instead*

$$[X_\mu, P_\nu] = g_{\mu\nu} - \frac{1}{\kappa^2}P_\mu P_\nu - \frac{1}{R^2}X_\mu X_\nu + \frac{1}{\kappa R}(X_\mu P_\nu + P_\mu X_\nu + M_{\mu\nu}). \quad (6)$$

The problem with this statement originates in the form Eq. (5) is written. Since we are dealing with a Lie algebra, the right hand side of that equation ought to be linear in the generators of the algebra. The equation is nevertheless generally written in that form because the corresponding generator, call it  $F$ , is central in the algebra. Writing the Jacobi identities with only a  $g_{\mu\nu}$  in the r.h.s. of (5) amounts to assuming that  $F$  remains central after the deformation, which is not true. Indeed, taking  $F$  into account, one finds from the Jacobi identity for the nested commutators  $[X_\rho, [X_\mu, P_\nu]]$ ,  $[P_\rho, [X_\mu, P_\nu]]$  that

$$[X_\rho, F] = \frac{1}{\kappa^2}P_\rho, \quad [P_\rho, F] = -\frac{1}{R^2}X_\rho, \quad (7)$$

where

$$[X_\mu, P_\nu] = g_{\mu\nu}F, \quad (8)$$

and  $F$  still commutes with the Lorentz sector. The resulting Lie algebra, given by Eqs. (1)–(4), (7) and (8) (all other commutators being zero) is a 3-SSR Lie algebra, with invariant velocity, mass and length scales set by  $c = 1$ ,  $\kappa$  and  $R$ , respectively — we denote it by  $\mathcal{G}_{\kappa, R}$  in what follows<sup>3</sup>.

Despite the obvious aesthetic and practical disadvantages, one can, in principle, choose to close the algebra non-linearly, maintaining  $F$  central, as in [8]. Given that the main motivation for doing so in the first place was the introduction of an additional invariant scale, and that this can be achieved, as shown above, without sacrificing linearity, there seems to be hardly any reason for making that choice. Additionally, one would then have to also supply a coalgebra structure, *i.e.*, a rule about how does the algebra act on tensor products of representations, satisfying reasonable *physical* requirements — without it, no composite physical systems can be considered and the usefulness of the deformation is drastically reduced. On the other hand, maintaining the deformation within the Lie algebra framework, has the added advantage that the standard coalgebra structure is available, namely, all generators act on tensor products as derivations.

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<sup>3</sup>The standard practice seems to be to exclude  $\hbar$  from the counting of invariant scales — we follow it here to avoid confusion.

### 3 The Stability Point of View

*Stable* Lie algebras are isomorphic to all Lie algebras with infinitesimally differing structure constants, *unstable* ones are not. When the structure constants of a particular Lie algebra used in physics involve experimentally determined quantities, *e.g.*, fundamental constants, then it is natural to seek a stable form of the algebra, in order to guarantee the robustness of the associated physics. This point of view has already a long history (see, for example, [2, 4, 9]) and can boast at least two (alas, *a posteriori*) predictions: (i) the algebra of Galilean kinematics is unstable — its stabilized form is relativistic kinematics (with the particular value of  $c$  fixed, of course, by experiment) and (ii) the infinite dimensional algebra of functions on classical phase space, with Lie product given by the Poisson bracket, is unstable, its stabilized form being quantum mechanics (with  $\hbar$  fixed by experiment). Faddeev has pointed out [3] that a similar relation exists between special and general relativity, the gravitational constant  $G$  being the deformation parameter in this case.

The mathematical aspects of the stability problem have been worked out in the classical contributions of Gerstenhaber [5], Nijenhuis and Richardson [10, 11], and others. There exists a beautiful cohomological description of the tangent space  $T\mathcal{L}_n$  to the space  $\mathcal{L}_n$  of all Lie algebras of a certain dimension, with directions leading to non-isomorphic Lie algebras being associated to non-trivial cocycles of a certain coboundary operator. This formulation, coupled with Whitehead's lemma, shows that semisimple Lie algebras are stable. In [9], the formalism was applied to the problem of determining the stable form of the Poincaré algebra, extended by the inclusion of the coordinates as generators. The result is, up to trivial rescalings and the additional freedom of certain signs, the 3-SSR Lie algebra  $\mathcal{G}_{\kappa,R}$  of the previous section<sup>4</sup>. Being semisimple, it is stable and hence no further non-trivial deformations are possible, and no new invariant scales can be introduced. As argued in [9], one may take the limit  $R \rightarrow \infty$  if one is interested in the kinematics in the tangent space, rather than the motions in the manifold itself. The resulting algebra  $\mathcal{G}_{\kappa,\infty}$  is then unstable, but this is just due to the fact that a flat manifold (the tangent space) is qualitatively different than a curved one. Viewed in another way, the resulting algebra is stable if one restricts to the subspace of  $\mathcal{L}_n$  of tangent space kinematical algebras.

### 4 Relation with Other Algebras

A number of non-commutative spacetime Lie algebras have been proposed over the years, but rarely did a full set of commutators (including momenta and the Lorentz sector) appear in these proposals. Additionally, non-linear deformations have appeared, so the task of comparing  $\mathcal{G}_{\kappa,R}$  with earlier approaches is not straight-forward. We undertake it in detail in our forthcoming article mentioned above, restricting ourselves to some brief remarks in this section — we emphasize that the list of references that follows is not exhaustive.

A non-commutative spacetime algebra, in which the coordinates generate rotations in a 5-dimensional space of constant negative curvature, was proposed almost sixty years ago by Snyder [12]. The  $x$ - $x$  relations there are identical to the ones in (3). The momenta commute (which corresponds to the  $R \rightarrow \infty$  limit mentioned above) but the  $P$ - $X$  relations contain non-linear terms. Soon thereafter, an article by Yang [13] pointed out that by recognizing the additional generator in the r.h.s. of the  $P$ - $X$  relations, as we propose in Sect. 2, one may render the algebra linear. His momenta cease to commute, in exactly the same way as in (4), and the algebra is, up to rescalings, the stable form  $\mathcal{G}_{\kappa,R}$  of [9]. We choose nevertheless [9] as our main reference, because there, as explained already, the physically sensible criterion of stability is proposed which, applied to the problem at hand, uniquely specifies its solution. The particular commutators in Yang's work were chosen so that the resulting algebra were  $O(1,5)$ , without any further justification or hint of uniqueness. A later work by Khrushchev and Leznov [6] claimed that

<sup>4</sup>A second stable algebra found in [9] seems less relevant physically — only the  $P$ - $X$ ,  $P$ - $F$ ,  $X$ - $F$  relations are deformed (*i.e.*, coordinates and momenta are still commutative), and the two invariant scales  $\kappa$  and  $R$  appear only as  $1/\kappa R$ , so that the whole deformation disappears when  $R \rightarrow \infty$ .

Yang's algebra could be further deformed, and introduced an additional invariant scale with dimensions of action. It can be easily shown though that the extra terms there can be reabsorbed by a linear redefinition mixing coordinates with momenta.

Apart from the above proposals, non-linear deformations have been introduced, as mentioned in the introduction. Since these involve a deformation of the coalgebra sector as well, a proper analysis and evaluation cannot be given here. It is nevertheless interesting to point out that several of the forms that have been proposed have been shown to be related by non-linear redefinitions of the generators [7]. The Lorentz-plus-coordinates sector can be brought in a linear form that coincides with the same sector of  $\mathcal{G}_{\kappa,R}$ , while the momenta commute. In other words, apart from the  $P$ - $X$  relations, the standard 2-SSR non-linear algebras (termed DSR1 and DSR2 in [7]) are isomorphic to  $\mathcal{G}_{\kappa,\infty}$ , while the linearized version of the 3-SSR of [8] is, as shown earlier,  $\mathcal{G}_{\kappa,R}$ . That the  $P$ - $X$  relations are different in DSR $n$  than those proposed here is hardly surprising given that in those works too, the extra generator  $F$  passes unnoticed. In any case, in those works the  $P$ - $X$  relations are derived from a Heisenberg double construction, in which the coalgebra sector plays a role — accordingly, we defer further comments to future work. We think that these brief remarks suffice to show that stability considerations are relevant to the problem of multiple-scale relativities and strongly suggest seeking a solution (or rather, using the one found long ago) within the standard Lie algebra framework.

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